Chapitre III : Déformations infinitésimales

I) La classe de Kodaira-Spencer

Soit $\mathcal{X} \xrightarrow{\pi} B$ une famille holomorphe de variétés analytiques complexes compactes. Comme π est une submersion, par le théorème d'inversion locale il existe pour t_0 fixé dans B :

- un voisinage Δ de t_0 dans B
- un recouvrement ouvert localement fini $(\mathcal{U}^\lambda)_{\lambda \in \Lambda}$ de $\pi^{-1}(\Delta)$
- des coordonnées holomorphes $t = (t_1, \dots, t_m) : \Delta \to \mathbb{C}^m$
- des coordonnées holomorphes $z^\lambda = (z_1^\lambda, \dots, z_n^\lambda, z_{n+1}^\lambda, \dots, z_{n+m}^\lambda) : \mathcal{U}^\alpha \longrightarrow \mathbb{C}^{n+m}$

tels que

$$\begin{tikzcd}
\mathcal{U}^\lambda \arrow[r, "z^\lambda"] \arrow[d, "\pi"'] & V^\lambda \subset \mathbb{C}^{n+m} & (z_1, \dots, z_n, z_{n+1}, \dots, z_{n+m}) \arrow[d, mapsto] \\
\Delta \arrow[r, "t"] & \mathbb{C}^m & (z_{n+1}, \dots, z_{n+m})
\end{tikzcd}$$

t centré en t_0

Par abus de notation, on notera $z_{n+i}^\lambda = t_i$.

Sur les intersections $\mathcal{U}^{\lambda\mu}$,

$$\begin{tikzcd}
z^\lambda \circ (z^\mu)^{-1} : z^\mu(\mathcal{U}^{\lambda\mu}) \arrow[r] \arrow[d, phantom, "\cap"] & z^\lambda(\mathcal{U}^{\lambda\mu}) \arrow[d, phantom, "\cap"] \\
V^\mu & V^\lambda
\end{tikzcd}$$

s'écrit $z_j^\lambda = f_j^{\lambda\mu}(z_1^\mu, \dots, z_n^\mu, t_1, \dots, t_m)$ $\quad 1 \leq j \leq n$

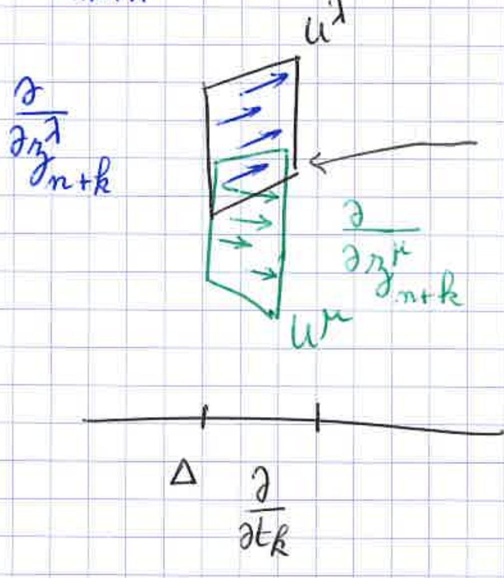
$z_{n+k}^\lambda = t_k = z_{n+k}^\mu$

En particulier, $X_0 = \pi^{-1}(b_0)$ est la variété obtenue en recollant $(U^\lambda \cap X_0)_{\lambda\in\Lambda}$ par $f^{\lambda\mu}(\cdot, \underbrace{0,\dots,0}_{m \text{ fois}})$

$$= (U_0^\lambda)_{\lambda\in\Lambda}$$

La donnée infinitésimale d'ordre 1 est donc la donnée des $\dfrac{\partial f_j^{\lambda\mu}}{\partial t_k}$ (car $f_j^{\lambda\mu}$ est holomorphe).

Le champ de vecteurs $\dfrac{\partial}{\partial t_k}$ sur Δ se relève sur U^λ en $\dfrac{\partial}{\partial z^\lambda_{n+k}}$.



Les deux relèvements ne coïncident pas nécessairement.

$$z_j^\lambda = f_j^{\lambda\mu}(z_i^\mu, \dots, z_n^\mu, t_1, \dots, t_m)$$

$$df^{\lambda\mu}\frac{\partial}{\partial z^\mu_{n+k}} = \sum_{j=1}^{n} \frac{\partial f_j^{\lambda\mu}}{\partial z^\mu_{n+k}} \frac{\partial}{\partial z_j^\lambda} + \frac{\partial}{\partial z^\lambda_{n+k}}$$

On définit

$$\Theta^{\lambda\mu}\cdot\frac{\partial}{\partial t} \overset{\text{def}}{=} \sum_{j=1}^{n} \frac{\partial f_j^{\lambda\mu}}{\partial t_k}\frac{\partial}{\partial z_j^\lambda} \in \mathcal{E}^1(U_0, TX_0)$$

Comme $f^{\lambda\mu}$ et $f^{\mu\lambda}$ sont inverses l'une de l'autre, $\Theta^{\lambda\mu}\cdot\dfrac{\partial}{\partial t_k}$ est alternée en (λ, μ).

Montrons que c'est un 1-cocycle à valeur dans TX_0.

$$f_i^{\lambda\nu}(z^\nu, t) = f_i^{\lambda\mu}(f_j^{\mu\nu}(z^\nu,t),t)$$

$$\theta^{\lambda\nu}\cdot\frac{\partial}{\partial t_k} = \sum_{j=1}^{n}\frac{\partial f_i^{\lambda\cancel{\mu}\nu}}{\partial t_k}\frac{\partial}{\partial z_i^\lambda} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial f_i^{\lambda\mu}}{\partial z_j^\mu}\frac{\partial f_j^{\mu\nu}}{\partial t_k}\frac{\partial}{\partial z_i^\lambda} + \sum_{i=1}^{n}\frac{\partial f_i^{\lambda\mu}}{\partial t_k}\frac{\partial}{\partial z_i^\lambda}$$

$$= \sum_{j=1}^{n}\frac{\partial f_j^{\mu\nu}}{\partial t_k}\underbrace{\sum_{i=1}^{n}\frac{\partial f_i^{\lambda\mu}}{\partial z_j^\mu}\frac{\partial}{\partial z_j^\lambda}}_{df^{\lambda\mu}\cdot\frac{\partial}{\partial z_j^\mu}} + \theta^{\lambda\mu}\cdot\frac{\partial}{\partial t_k}$$

$$= df^{\lambda\mu}\cdot\theta^{\mu\nu}\cdot\frac{\partial}{\partial t_k} + \theta^{\lambda\mu}\cdot\frac{\partial}{\partial t_k}$$

Donc $(\theta^{\lambda\mu})_{(\lambda,\mu)\in\Lambda^2} \in Z^1(\mathcal{U}_0, TX_0)$.

Lemme

- La classe $[\theta^{\lambda\mu}\cdot\frac{\partial}{\partial t_k}] \in \check{H}^1(X_0, TX_0)$ est indépendante des choix de recouvrement et de carte. On la note $\frac{\partial X_t}{\partial t_k}$.
- On a une application linéaire

$$\begin{aligned} T_{t_0}B &\longrightarrow \check{H}^1(X_0, TX_0) \\ \frac{\partial}{\partial t_k} &\longmapsto \frac{\partial X_t}{\partial t_k} \end{aligned}$$

C'est le morphisme connectant dans la suite exacte longue déduite de la suite exacte courte de faisceaux sur X_0.

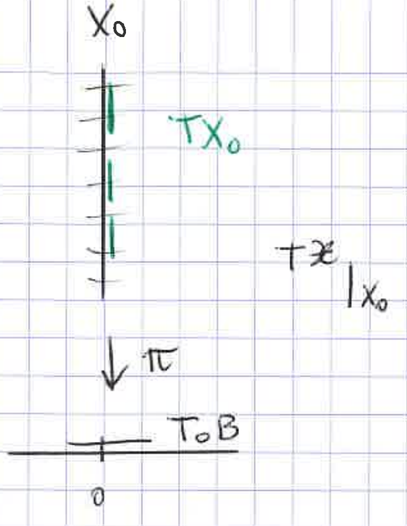
Plusieurs faisceaux correspondant à un fibré.

$$0 \longrightarrow TX_0 \longrightarrow T\mathcal{X}_{|X_0} \xrightarrow{d\pi} \pi^* T_0 B \longrightarrow 0$$

ou plutôt

$$0 \longrightarrow \mathcal{T}X_0 \longrightarrow \mathcal{T}\mathcal{X}_{|X_0} \longrightarrow \pi^*\mathcal{T}_0 B \longrightarrow 0$$

faisceau des germes de sections holomorphes de TX_0



TX_0 est le noyau de $d\pi$

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X_0, \mathcal{T}X_0) \arrow[r] & \Gamma(X_0, \mathcal{T}\mathcal{X}_{|X_0}) \arrow[r] & T_{b_0}B \arrow[dll, "\text{morphisme connectant}"] \\
 & \check{H}^1(X_0, \mathcal{T}X_0) & &
\end{tikzcd}$$

↑ champs de vecteur globaux sur X_0

Proposition → famille…

Si $\pi : \mathcal{X} \longrightarrow B$ est localement sur B holomorphiquement triviale au voisinage de b_0 alors $\forall \frac{\partial}{\partial t} \in T_{b_0}B$, $\frac{\partial X_t}{\partial t} = 0$.

dérivée d'une fct est nulle

Démo

L'hypothèse indique qu'on peut choisir $(f^{\lambda\mu}(z^\mu, t))_{(\lambda,\mu)\in\Lambda^2}$ indépendamment de t.

$$\begin{tikzcd}
\mathcal{X}_{|\Delta} \arrow[r, "\text{biholo}", "\sim"'] \arrow[d] & X_0 \times \Delta \arrow[d] \\
b_0 \in \Delta \arrow[r, equal] & \Delta
\end{tikzcd}$$

Théorème (de Fisher - Grauert)

Si $\pi : \mathcal{X} \longrightarrow B$ est une famille holomorphe de variétés complexes compactes et si $\forall (b, b') \in B^2$ $X_b = \pi^{-1}(b)$ et $X_{b'}$ sont biholomorphes alors π est localement sur B holomorphiquement triviale.

On a vu que localement sur $\mathcal{X}$, c'est tjrs vrai.

NB $KS_0 : T_b B \longrightarrow \check{H}^1(X_0, \mathcal{T}X_0)$ s'appelle application de Kodaira-Spencer.

Théorème

Si $\pi : \mathcal{X} \longrightarrow B$ est une famille holomorphe ...
et si $\forall b \in B$

$$KS_b : T_b B \longrightarrow \check{H}^1(X_b, \mathcal{T}X_b) \text{ est nulle}$$

et si $b \longmapsto \dim \check{H}^1(X_b, \mathcal{T}X_b)$ est une fonction constante, alors la famille est localement sur B holomorphiquement triviale.

dérivée partout nulle $\not\Rightarrow$ cte.

Condition non satisfaite pour Hirzebruch.

12 février

II In terms of Dolbeault cohomology

Let $\pi : \mathcal{X} \longrightarrow B$ be an analytic family of compact complex manifolds.

If B is small enough, π is diffeomorphic to a product.

differentiable

$$\begin{tikzcd}
\mathbb{C} & \mathcal{X} \arrow[l, "z_j^{\lambda}"'] \arrow[d, "\pi"'] & M \times B \arrow[l, "\tau"'] \arrow[d, "pr_2"] \\
 & B \arrow[r, equal] & B
\end{tikzcd}$$

$M = (X_0)_{diff}$

$0 \in B$

not a product → monodromy

Let (z_i^λ, t_j) be local holomorphic coordinates on $\mathcal{X}$, as before.

Choose $z_j^\lambda(\tau(m,0))\cancel{=\tau^{-1}(z^\lambda_{j|X_0})} =: x_j^\lambda + i y_j^\lambda$ as smooth coordinates on $(X_0)_{diff} = M$. on $\tau^{-1}(U^\lambda)_{|t=0}$.

$$(m,t) \longmapsto z_j^\lambda(\tau(m,t)) \quad \text{smooth functions on } M\times B$$
$$\parallel$$
$$\zeta_j^\lambda(m,t) \quad \text{giving smooth coordinates on each } X_t.$$

They fullfill the relation
$$\zeta_j^\lambda(m,t) = f_j^{\lambda\mu}(\zeta_k^\mu(m,t),t) \qquad (*)$$

Let Θ_0 be the sheaf of germs of biholomorphic sections of the holomorphic tangent bundle TX_0 of X_0.
Dolbeault complex gives a resolution of Θ_0

$$\begin{tikzcd}
0 \arrow[r] & \Theta_0 \arrow[r] & \mathcal{A}^0(TX_0) \arrow[rr, "\bar\partial"] \arrow[dr, "\bar\partial"'] & & \mathcal{A}^{0,1}(TX_0) \arrow[r] & \cdots \\
& & 0 \arrow[r] & \partial\mathcal{A}^0(TX_0) \arrow[ur] \arrow[r] & 0 &
\end{tikzcd}$$

~~and hence~~,

and hence, because the sheaves $\mathcal{A}^{0,q}(TX_0)$ are fine, the connecting homomorphism gives an isomorphism

$$\begin{tikzcd}
H^{0,1}_{\bar\partial}(X_0,TX_0) := \dfrac{\Gamma(X_0,\bar\partial\mathcal{A}^0(TX_0))}{\bar\partial\,\Gamma(X_0,\mathcal{A}^0(TX_0))} \arrow[r, "\delta", "\sim"'] & \check{H}^1(X_0,\Theta_0) \\
\rotatebox{90}{$\in$} & \rotatebox{-90}{$\in$} \\
& KS\left(\frac{\partial}{\partial t}\right) \text{ for } \frac{\partial}{\partial t}\in T_0B \\
& \parallel \\
\eta \arrow[r, mapsto, "\delta"] & \frac{\partial X_t}{\partial t}
\end{tikzcd}$$

$\uparrow$
Dolbeault cohomology

η defined p. 65

$$\frac{\partial X_t}{\partial t} = \left(\theta^{\lambda\mu}\right) = \left(\sum_{i=1}^{n} \frac{\partial f_i^{\lambda\mu}}{\partial t} \frac{\partial}{\partial z_i^\lambda}\right)$$

using (*) $$= \left(\sum_{j=1}^{n} \frac{\partial \zeta_j^\lambda}{\partial t}\frac{\partial}{\partial z_j^\lambda} - \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial f_j^{\lambda\mu}}{\partial z_k^\mu}\frac{\partial \zeta_k^\mu}{\partial t}\frac{\partial}{\partial z_j^\lambda}\right)$$

$$= \left(\sum_{j=1}^{n} \frac{\partial \zeta_j^\lambda}{\partial t}\frac{\partial}{\partial z_j^\lambda} - \sum_{k=1}^{n} \frac{\partial \zeta_k^\mu}{\partial t}\frac{\partial}{\partial z_k^\mu}\right)$$

locally $\bar\partial$ exact
not necessarily
in $\bar\partial\Gamma(X_0, \mathcal{A}^0(TX_0))$

Hence $\bar\partial\left(\sum_{j=1}^{n} \frac{\partial \zeta_j^\lambda}{\partial t}\frac{\partial}{\partial z_j^\lambda}\right)_{\lambda\in\Lambda}$ agree on $\mathcal{U}^\lambda \cap \mathcal{U}^\mu$ as $\bar\partial\left(\theta^{\lambda\mu}\right) = 0$.

This defines an element η in $\Gamma(X_0, \bar\partial\mathcal{A}^0(TX_0))$ such that $\delta\eta = \frac{\partial X_t}{\partial t}$.

III Description of neighbouring complex structure

Choose any holomorphic coordinates (z_i) on X_0.

As $\zeta_i^\lambda(m, 0)$ is holomorphic in (z_i),

$\det\left(\frac{\partial \zeta_i^\lambda}{\partial z_i}\right) \neq 0$ for $t = 0$, $|t| \ll 1$.

$$A_{ij}^\lambda := \frac{\partial \zeta_i^\lambda}{\partial z_j} = \sum_k \frac{\partial f_i^{\lambda\mu}}{\partial \zeta_k^\mu}\frac{\partial \zeta_k^\mu}{\partial z_j} + \frac{\partial f_i^{\lambda\mu}}{\partial t}\underbrace{\frac{\partial t}{\partial z_j}}_{=0}$$

$$= A_{ik}^{\lambda\mu} A_{kj}^\mu$$

where $A_{ik}^{\lambda\mu} = \frac{\partial f_i^{\lambda\mu}}{\partial \zeta_k^\mu}$ and $\left(A_{ij}^\lambda\right)_{i,j}$ is invertible.

Denote B_{ji}^λ ~~invertible~~ its inverse.

$$\bar\partial \zeta_i^\lambda = \sum_j \frac{\partial \zeta_i^\lambda}{\partial \bar z_j} d\bar z_j = \sum\left(\frac{\partial f_i^\lambda}{\partial z_k^\mu}\frac{\partial \zeta_k^\mu}{\partial \bar z_j} + \underbrace{\frac{\partial f_i^{\lambda\mu}}{\partial \bar z_k^\mu}}_{=0}\overline{\frac{\partial \zeta_k^\mu}{\partial z_j}}\right) d\bar z_j$$

because

$(z^\mu, t) \longmapsto f_i^{\lambda\mu}(z^\mu, t)$ are holom.

NB $\sum_j A_{ij}^\lambda B_{jk}^\lambda = \delta_{ik}$ (Kronecker delta)

$$\bar\partial \zeta_i^\lambda = \sum_k A^{\lambda\mu}_{ik} \bar\partial \zeta_k^\mu$$

Define $\varphi_j := \sum_i B^\lambda_{ji} \bar\partial \zeta_i^\lambda = \sum_k B^\mu_{jk} \bar\partial \zeta_k^\mu$

$\varphi(t) := \sum_j \varphi_j \frac{\partial}{\partial z_j}$ is well defined on $U^\lambda \cap U^\mu$

and is independent of the choice of (z_j).

It is a global $(0,1)$-form on X_0 with values in TX_0.

$$A^\lambda_{ij} \varphi_j = \bar\partial \zeta_i^\lambda = \frac{\partial \zeta_i^\lambda}{\partial z_j} \varphi_j$$

Hence the $\zeta_i^\lambda(m,t)$ satisfy the equation $(\bar\partial - \varphi)\zeta_i^\lambda = 0$.

on M.

Proposition

- $\varphi(0) = 0$
- A differentiable function f on M is holomorphic for X_t if and only if $(\bar\partial - \varphi) f = 0$
- $\frac{\partial X_t}{\partial t}\Big|_{t=0}$ is represented in Dolbeault cohomology by $+\frac{\partial \varphi}{\partial t}\Big|_{t=0}$.

$\Big($ For $t=0$, $A^\lambda_{ij} = \delta_{ij}$, $\varphi = \sum_j \varphi_j \frac{\partial}{\partial z_j} = \sum_j \bar\partial \zeta_j^\lambda \frac{\partial}{\partial z_j}$

$\frac{\partial \varphi}{\partial t} = \bar\partial\left(\sum_j \frac{\partial \zeta_i^\lambda}{\partial t} \frac{\partial}{\partial z_j}\right) \Big)$

- $\bar\partial_{X_0} \varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$

Proof (of the last point)

Recall the bracket of vector fields

$u = \sum u_i \frac{\partial}{\partial z_i}$, $v = \sum v_i \frac{\partial}{\partial z_i}$

$$[u,v] \overset{\text{def}}{=} \sum \left(u_i \cdot \frac{\partial v_j}{\partial z_i} - v \cdot \frac{\partial u_j}{\partial z_i} \right) \frac{\partial}{\partial z_j}$$

This is extendend on $(0,q)$- forms with values in TX_0, using the wedge product instead of the dot product.

$$\varphi = \varphi_j \frac{\partial}{\partial z_j} \qquad \text{(Einstein summation convention).}$$

φ_j is a $(0,1)$- form

$$[\varphi,\varphi] = \left(\varphi_j \wedge \frac{\partial}{\partial z_j} \varphi_k - (-1)^{1\times 1} \varphi_j \wedge \frac{\partial}{\partial_j} \varphi_k \right) \frac{\partial}{\partial z_k}$$

$$\left([\varphi,\psi] = \left(\varphi_j \wedge \frac{\partial}{\partial z_j} \psi_k - (-1)^{\deg\varphi \deg\psi} \psi_j \wedge \frac{\partial}{\partial z_j} \varphi_k \right) \frac{\partial}{\partial z_k} \right)$$

(where $\psi = \sum \psi_k \frac{\partial}{\partial z_k}$, ψ_k is a $(0,\ell)$-form, φ_j is a $(0,q)$-form)

$$\left(\begin{array}{l} \varphi_k = \varphi_{kI} \, d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{iq} \\ \frac{\partial}{\partial z_j} \varphi_k = \frac{\partial \varphi_{kI}}{\partial z_j} \, d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i\ell} . \end{array} \right)$$

$$[\varphi,\varphi] = 2\left(\varphi_j \wedge \frac{\partial \varphi_\ell}{\partial z_j} \right) \frac{\partial}{\partial z_k}$$

$$\bar{\partial} \zeta_i^\lambda = \varphi_j \frac{\partial}{\partial z_j} \zeta_i^\lambda$$

$$\bar{0} = \bar{\partial}\bar{\partial} \zeta_i^\lambda = \bar{\partial} \varphi_j \frac{\partial \zeta_i^\lambda}{\partial z_j} - \varphi_j \frac{\partial}{\partial z_j} \overbrace{\bar{\partial} \zeta_i^\lambda}^{\varphi_k \frac{\partial \zeta_i^\lambda}{\partial z_k}}$$

(Leibniz rule $\bar{\partial}(a \wedge b) = \bar{\partial} a \wedge b - (-1)^{\deg a} a \wedge \bar{\partial} b$.)

$$= \bar{\partial} \varphi_j \frac{\partial \zeta_i^\lambda}{\partial z_j} - \varphi_j \frac{\partial \varphi_k}{\partial z_j} \frac{\partial \zeta_j^\lambda}{\partial z_k} - \underbrace{\varphi_j \wedge \varphi_k}_{\text{anti-sym.}} \underbrace{\frac{\partial^2 \zeta^\lambda}{\partial z_i \partial z_k}}_{\text{symmetric in } (i,k)}$$

Hence $\bar{\partial} \varphi = \frac{1}{2} [\varphi,\varphi]$.